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Some Remarks on the Notion of Contraction of Lie Group Representations

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Abstract. In the series of papers [1-4], L. Barker developed a general notion of convergence for sequences of Hilbert spaces and related objects (vectors, operators...). In this paper, we remark that Barker's convergence for sequences of operators provides a notion of contraction of Lie group (unitary) representations and we compare it to the usual one introduced by J. Mickelsson and J. Niederle. This allows us to illustrate Barker's convergence of operators by various examples taken from contraction theory.

1. Introduction

In the pioneering paper [19], Inönü and Wigner introduced the notion of contraction of Lie groups and Lie group representations on physical grounds: If two physical theories are related in a limiting process, then the associated invariance groups and their representations should be also related in a limiting process called contraction. For instance, the Galilei group is a contraction of the Poincaré group [19].

Contractions of Lie algebras, Lie groups and their representations have been studied by many authors and continue to be a subject of active research, see for instance the papers [17], [18] and their references.

In fact, the systematic study of contractions of Lie group representations began with the work of Mickelsson and Niederle. In [22], a proper definition of the contraction of unitary representations of Lie groups was given for the first time and was illustrated by various examples, including contractions of the principal series representations of $SO_0(n+1,1)$ to the non-zero mass representations of the Euclidean group $\mathbb{R}^{n+1} \rtimes SO(n+1)$ and to the positive mass-squared representations of the Poincaré group $\mathbb{R}^{n+1} \rtimes SO_0(n,1)$. More generally, in [16], Dooley and Rice established a contraction of the principal series representations of a semi-simple Lie group to some unitary irreducible representations of its Cartan motion group.

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In [14] and [13], contractions of representations were interpreted in the setting of the Kirillov-Kostant method of orbits [20] and, in [13] and [5]-[10], Berezin and Weyl quantization maps were used in order to obtain contraction results.

Recently, in the series of papers [1]-[4], L. Barker developed a general theory of convergence for a sequence $(\mathcal{H}_n)$ of Hilbert spaces in order to describe continuum quantum systems as limits of discrete quantum systems. This theory includes a notion of convergence for a sequence $f_n \in \mathcal{H}_n$ and for a sequence (A_n) where A_n is a bounded operator on $\mathcal{H}_n$ for each integer $n > 0$.

Although this is not said explicitly in [3], it appears that Barker's convergence of operators immediately gives a new notion of contraction of Lie group unitary representations. The main goal of the present paper is then to compare this notion to the usual one introduced in [22].

This paper is organized as follows. In Section 2, we recall some basic facts about contractions and, in Section 3, we outline Barker's theory. In Section 4, we establish our main results. We compare the notion of contraction of Lie group unitary representations which derives from Barker's theory with that of Mickelsson and Niederle. In particular, we show that a contraction of Lie group representations in the sense of [22] is also a contraction in the sense of Barker's theory, the converse being true under some additional assumptions. In Section 5, we give some examples of contractions and we mention some open questions.

2. Generalities on Contractions

In this section, we review some basic facts on contractions. The material of this section is essentially taken from [22] (see also [8]). We begin by recalling the definitions of a Lie algebra contraction and a Lie group contraction.

Let G and H be two real Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We assume that $\mathfrak{g}$ and $\mathfrak{h}$ have the same dimension and we denote by $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_0$ the Lie brackets on $\mathfrak{g}$ and $\mathfrak{h}$, respectively.

Definition 2.1. A contraction of $\mathfrak{g}$ to $\mathfrak{h}$ is a family $(C_r)_{r\in]0,1]}$ of linear isomorphisms from $\mathfrak{g}$ onto $\mathfrak{h}$ such that

$$\lim_{r\to 0} C_r^{-1}\left[C_r\left(X\right), C_r\left(Y\right)\right]_1 = [X,Y]_0$$

for all X and Y in $\mathfrak{h}$.

Definition 2.2. A contraction of G to H is a family $(c_r : V \to G)_{r\in]0,1]}$ of smooth maps defined on the same neighborhood V of the identity element e_H of H satisfying the properties:

(1) For each $r \in]0,1]$, c_r maps e_H to the identity element e_G of G;
(2) There exists an open neighborhood W of e_G such that c_r is a diffeomorphism of $c_r^{-1}(W^2)$ onto W^2 for each $r \in]0,1]$;

(3) For each $x \in V$ there is $r_x \in]0,1]$ such that $c_r(x) \in W$ for $r < r_x$;
(4) For all x, y in V, we have

$$\lim_{r \to 0} c_r^{-1}\left(c_r(x).\, c_r(y)\right) = xy.$$

Because of (2) and (3), the expression that is taken to the limit in (4) is well defined for $r < r_x, r_y$. If the family $(c_r)_{r \in]0,1]}$ is a contraction of G to H then the family $(dc_r(e_H))_{r \in]0,1]}$ is a contraction of $\mathfrak{g}$ to $\mathfrak{h}$. Conversely, if $(C_r)_{r \in]0,1]}$ is a contraction of $\mathfrak{g}$ to $\mathfrak{h}$ such that the family $(\|C_r\|_{\mathrm{op}},\, r \in]0,1])$ is bounded then by adapting arguments of the proof of Theorem 2.15.4 in [24] one can show that the family $(C_r)_{r \in]0,1]}$ exponentiates to a contraction $(c_r = \exp_G \circ C_r \circ \exp_H^{-1})_{r \in]0,1]}$ of G to H.

Now, we fix a contraction $(c_r)_{r \in]0,1]}$ of G to H as in Definition 2.2. For each integer $n > 0$ let π_n be a unitary representation of G on a Hilbert space $\mathcal{H}_n$. Let ρ be a unitary representation of H in a Hilbert space $\mathcal{H}$. We denote by $\langle \cdot, \cdot \rangle_n$ and $\langle \cdot, \cdot \rangle$ the scalar products on $\mathcal{H}_n$ and $\mathcal{H}$ and by $\| \cdot \|_n$ and $\| \cdot \|$ the corresponding norms.

Definition 2.3. We say that the representation ρ is a MN-contraction of the sequence (π_n) if there exists a sequence $r(n) \in]0,1]$ with limit 0, a sequence of unitary operators $A_n : \mathcal{H}_n \to \mathcal{H}$ and a dense subspace D of $\mathcal{H}$ satisfying the following properties:

(1) For each $f \in D$ there exists an integer $n_0 > 0$ such that for each $n \geq n_0$ we have $f \in A_n(\mathcal{H}_n)$;
(2) For each $f \in D$ and $h \in V$, we have

$$\lim_{n \to +\infty} \|A_n\, \pi_n\left(c_{r(n)}\left(h\right)\right) A_n^{-1}\, f - \rho\left(h\right) f\| = 0.$$

Note that the expression that is taken to the limit in Definition 2.3 is well-defined for $n \geq n_0$.

Remark 2.4. Let f_1 and f_2 in D. Clearly, since the operators A_n are unitary, Property (2) of Definition 2.3 implies that

$$\lim_{n \to +\infty} \langle \pi_n\left(c_{r(n)}(h)\right) A_n^{-1} f_1\,,\, A_n^{-1} f_2 \rangle_n = \langle \rho\left(h\right) f_1\,,\, f_2 \rangle \tag{2.1}$$

for each $h \in V$. Conversely, assume that (2.1) holds for each $f_1, f_2 \in D$ and each $h \in V$. Let $f \in D$ and $h \in V$. Then we immediately see that $(A_n\, \pi_n\left(c_{r(n)}\left(h\right)\right) A_n^{-1}\, f)$ converges weakly to $\rho\left(h\right) f$ in $\mathcal{H}$. Since $\|A_n\, \pi_n\left(c_{r(n)}\left(h\right)\right) A_n^{-1}\, f\| = 1 = \|\rho\left(h\right) f\|$, we have that $(A_n\, \pi_n\left(c_{r(n)}\left(h\right)\right) A_n^{-1}\, f)$ converges strongly to $\rho\left(h\right) f$ in $\mathcal{H}$.

Remark 2.5. Here we mention two important particular cases of MN-contractions.

1) The case when $A_n(\mathcal{H}_n) = \mathcal{H}$ for each n. In that case, we can assume that $D = \mathcal{H}$ in Definition 2.3. Indeed, if Property (2) of Definition 2.3 holds for each $f \in D$ then Property (2) also holds for each

$f \in \mathcal{H}$. Moreover, if (f_p) is an orthonormal basis of $\mathcal{H}$, then for each n, $f_p^n = A_n^{-1} f_p$ is an orthonormal basis for $\mathcal{H}_n$ and, according to Remark 2.4, we see that Property (2) of Definition 2.3 is equivalent to the fact that

$$\lim_{n\to+\infty} \langle \pi_n\left(c_{r(n)}(h)\right) f_p^n\, ,\, f_q^n\rangle_n = \langle \rho\,(h) f_p\, ,\, f_q\rangle \tag{2.2}$$

for each $h \in V$ and each p, q. An example of such a situation is given in Section 5.

2) The case when $\mathcal{H}_n$ is finite-dimensional for each n. See in particular [15], [23], [6] and [9].

Remark 2.6. In the setting of Definition 2.3, note that, since the operators A_n are unitary, $A_n(\mathcal{H}_n)$ is a closed subspace of $\mathcal{H}$ for each n. Let R_n be the operator from $\mathcal{H}$ to $\mathcal{H}_n$ defined by $R_n f = A_n^{-1} f$ if $f \in A_n(\mathcal{H}_n)$ and $R_n f = 0$ if f is orthogonal to $A_n(\mathcal{H}_n)$. Then we have

$$\lim_{n\to+\infty} \langle R_n f_1\, ,\, R_n f_2\rangle_n = \langle f_1\, ,\, f_2\rangle$$

for each f_1, $f_2 \in D$.

In the literature, we can also find the following notion of contraction of Lie group unitary representations which is weaker than MN-contraction (see for instance [12]).

Definition 2.7. If, in Definition 2.3, we replace the condition that the operators A_n are unitary by the condition that the operators A_n are injective, continuous, satisfying $\lim_{n\to+\infty} \|A_n^{-1} f\|_n = \|f\|$ for each $f \in D$, then we say that ρ is a nuMN-contraction of (π_n).

3. Barker's Theory

In this section, we outline Barker's theory. See [1], [3] and, for applications to Physics, [2] and [4].

As in Section 2, we consider a sequence $(\mathcal{H}_n)$ of Hilbert spaces, a Hilbert space $\mathcal{H}$ and a dense subspace D of $\mathcal{H}$. Let $R_n : D \to \mathcal{H}_n$ be a sequence of linear maps satisfying

$$\langle f\, ,\, g\rangle = \lim_{n\to+\infty} \langle R_n f\, ,\, R_n g\rangle_n \tag{3.1}$$

for each f, $g \in D$. The family $(\mathcal{H}_n, R_n)$ is then called an inductive resolution of $\mathcal{H}$.

Definition 3.1 ([1])**.** Let $f \in \mathcal{H}$ and, for each n, $f_n \in \mathcal{H}_n$. We say that the sequence (f_n) converges to f if the sequence $(\|f_n\|_n)$ is bounded and

$$\langle g\, ,\, f\rangle = \lim_{n\to+\infty} \langle R_n g\, ,\, f_n\rangle_n$$

for each $g \in D$. We call f the limit of (f_n) and we write $f = \lim_{n\to+\infty} f_n$.

Clearly, we have $f = \lim_{n\to+\infty} R_n f$ for each $f \in D$. Moreover, it was shown in [1] that any $f \in \mathcal{H}$ is the limit of a sequence $f_n \in \mathcal{H}_n$ such that $\|f_n\|_n = \|f\|$ for each n.

The following result will be needed in Section 4.

Proposition 3.2 ([1]). *Let $(e_p)_p$ be an orthonormal basis of $\mathcal{H}$. Then we can choose an orthonormal basis $(e^n_p)_{0\le p<p_n}$ of $\mathcal{H}_n$ for each n (here $p_n \in \mathbb{N}\cup(+\infty)$) in such a way that $\lim_{n\to+\infty} e^n_p = e_p$ for each p. Moreover, if we consider a sequence $f_n \in \mathcal{H}_n$ and an element $f \in \mathcal{H}$ and we write $f = \sum_p a_p e_p$ and $f = \sum_p a^n_p e^n_p$ with the understanding that $a^n_p = 0$ for $p \ge p_n$, then (f_n) converges to f if and only if $(\|f_n\|_n)$ is bounded and $\lim_{n\to+\infty} a^n_p = a_p$ for each p.*

Definition 3.3 ([3]). Let B be a bounded operator on $\mathcal{H}$ and, for each n, let B_n be a bounded operator on $\mathcal{H}_n$. We say that the sequence (B_n) converges to B if the sequence $(\|B_n\|_{\mathrm{op}})$ is bounded and if for each $f \in \mathcal{H}$ and each sequence $f_n \in \mathcal{H}_n$ with limit f, we have $Bf = \lim_{n\to+\infty} B_n f_n$.

Now we deduce from Definition 3.3 a new notion of contraction for Lie group representations. As in Section 2, we consider two real Lie groups G and H, a group contraction $(c_r)_{r\in]0,1]}$ of G to H and a neighborhood V of e_H as in Definition 2.2. Let ρ be a unitary representation of H on a Hilbert space $\mathcal{H}$ and, for each n, let π_n be a unitary representation of G on a Hilbert space $\mathcal{H}_n$.

Definition 3.4. We say that ρ is a B-contraction of the sequence (π_n) if there exists a dense space D of $\mathcal{H}$, an inductive resolution $(\mathcal{H}_n, R_n : D \to \mathcal{H}_n)$ of $\mathcal{H}$ and a sequence $r(n) \in]0,1]$ with limit 0 such that the sequence $\pi_n(c_{r(n)}(h))$ converges to $\rho(h)$ for each $h \in V$. If moreover, each operator R_n can be extended to a continuous operator $\tilde{R}_n$ from $\mathcal{H}$ onto $\mathcal{H}_n$ such that $\tilde{R}_n|_{(\mathrm{Ker}\,\tilde{R}_n)^\perp}$ is unitary, then we say that ρ is a uB-contraction of (π_n).

4. Comparison Between Different Notions of Contraction

In this section, we compare the notions of contractions of Lie group representations introduced in the previous sections. First, we compare the notion of MN-contraction to that of uB-contraction. These two notions are particularly adapted to the unitary setting.

As in the previous sections, we consider two real Lie groups G and H. We assume that there exists a group contraction $(c_r)_{r\in]0,1]}$ of G to H. We also consider a unitary representation ρ of H on a Hilbert space $\mathcal{H}$ and, for each n, a unitary representation π_n of G on a Hilbert space $\mathcal{H}_n$.

Proposition 4.1. *If ρ is a MN-contraction of (π_n) then ρ is a uB-contraction of (π_n).*

Proof. If ρ is a MN-contraction of (π_n) then we can define the operators R_n as in Remark 2.6. Thus $(\mathcal{H}_n, R_n)$ is an inductive resolution of $\mathcal{H}$.

Let $D \subset \mathcal{H}$, $V \subset H$ and $(r(n))$ as in Definition 2.3. We fix $h \in V$ and in order to simplify the notation we put $B_n := \pi_n(c_{r(n)}(h))$ and $B := \rho(h)$.

Fix $f \in \mathcal{H}$. Let $f_n \in \mathcal{H}_n$ be a sequence which converges to f. We have to prove that $(B_n f_n)$ converges to Bf. First, we note that the norms $\|B_n f_n\|_n = \|f_n\|_n$ are bounded.

Let (e_p) be an orthonormal basis of $\mathcal{H}$ consisting of elements of D. For each n, let (e_n^p) be an orthonormal basis of $\mathcal{H}_n$ as in Proposition 3.2, that is, (e_n^p) converges to e_p for each p. We can write as in Proposition 3.2

$$f_n = \sum_{0 \leq q < p_n} a_q^n e_q^n, \qquad f = \sum_q a_q e_q.$$

We also put

$$c_{pq}^n := \langle B_n e_p^n, e_q^n \rangle_n, \qquad c_{pq} := \langle B e_p, e_q \rangle.$$

The proof is now divided into four steps.

1) Firstly, we note that $\lim_{n \to +\infty} \|A_n^{-1} e_p - e_p^n\|_n = 0$ for each p. Indeed, we have

$$\|A_n^{-1} e_p - e_p^n\|_n^2 = \|A_n^{-1} e_p\|_n^2 + \|e_p^n\|_n^2 - 2\,\mathrm{Re}\langle A_n^{-1} e_p, e_p^n \rangle_n$$

where $\|A_n^{-1} e_p\|_n^2 = \|e_p^n\|_n^2 = 1$ and $\langle A_n^{-1} e_p, e_p^n \rangle_n = \langle R_n e_p, e_p^n \rangle_n$ converges to $\langle e_p, e_p \rangle = 1$ as $n \to +\infty$ because (e_p^n) converges to e_p.

2) Secondly, we show that $\lim_{n \to +\infty} c_{pq}^n = c_{pq}$ for each p and q. This can be done as follows. We have

$$\begin{aligned}
&\langle B_n e_p^n, e_q^n \rangle_n - \langle B_n A_n^{-1} e_p, A_n^{-1} e_q \rangle_n \\
&\quad = \langle B_n(e_p^n - A_n^{-1} e_p), A_n^{-1} e_q \rangle_n + \langle B_n A_n^{-1} e_q, e_q^n - A_n^{-1} e_q \rangle_n \\
&\quad + \langle B_n(e_p^n - A_n^{-1} e_p), e_q^n - A_n^{-1} e_q \rangle_n.
\end{aligned}$$

Since the operators B_n and A_n are unitary, this implies that

$$\begin{aligned}
&|\langle B_n e_p^n, e_q^n \rangle_n - \langle B_n A_n^{-1} e_p, A_n^{-1} e_q \rangle_n| \\
&\quad \leq \|e_p^n - A_n^{-1} e_p\|_n + \|e_q^n - A_n^{-1} e_q\|_n \\
&\quad + \|e_p^n - A_n^{-1} e_p\|_n \|e_q^n - A_n^{-1} e_q\|_n
\end{aligned}$$

By using Point 1), we then obtain

$$\lim_{n \to +\infty} (\langle B_n e_p^n, e_q^n \rangle_n - \langle B_n A_n^{-1} e_p, A_n^{-1} e_q \rangle_n) = 0.$$

On the other hand, by Remark 2.5, we have that

$$\lim_{n \to +\infty} \langle B_n A_n^{-1} e_p, A_n^{-1} e_q \rangle_n = \langle B e_p, e_q \rangle = c_{pq}.$$

This gives the result.

3) Now we will show that

$$\lim_{n\to+\infty}\langle B_nf_n,e_p^n\rangle_n=\langle Bf,e_p\rangle \tag{4.1}$$

for each p. To this aim, we write

$$\begin{aligned}\langle B_nf_n,e_p^n\rangle_n&=\langle f_n,B_n^{-1}e_p^n\rangle_n=\sum_q\langle f_n,e_q^n\rangle_n\langle e_q^n,B_n^{-1}e_p^n\rangle_n\\&=\sum_q\langle f_n,e_q^n\rangle_n\langle B_ne_q^n,e_p^n\rangle_n=\sum_q a_q^nc_{qp}^n\end{aligned}$$

and, similarly,

$$\langle Bf,e_p\rangle=\sum_q a_qc_{qp}.$$

Now we fix $\varepsilon>0$. Let $M=1+\|f\|+\sup_n\|f_n\|_n$. Note that

$$1=\|B^{-1}e_p\|^2=\sum_q|\langle Be_q,e_p\rangle|^2=\sum_q|c_{qp}|^2.$$

Choose q_0 so that $\sum_{q>q_0}|c_{qp}|^2<\varepsilon^2/9M^2$. Thus we have

$$1-\varepsilon^2/9M^2<\sum_{q\le q_0}|c_{qp}|^2$$

and using Point 2) we see that there exists n_1 so that

$$1-\varepsilon^2/9M^2<\sum_{q\le q_0}|c_{qp}^n|^2$$

and hence $\sum_{q>q_0}|c_{qp}^n|^2<\varepsilon^2/9M^2$ for each $n\ge n_1$. Applying the Cauchy-Schwarz inequality, we then obtain

$$\begin{aligned}\sum_{q>q_0}|a_q^nc_{qp}^n|&\le\left(\sum_{q>q_0}|a_q^n|^2\right)^{1/2}\left(\sum_{q>q_0}|c_{qp}^n|^2\right)^{1/2}\\&\le\|f_n\|_n\,\varepsilon/3M\le\varepsilon/3\end{aligned}$$

for each $n\ge n_1$. Similarly, we have

$$\sum_{q>q_0}|a_qc_{qp}|\le\|f\|\,\varepsilon/3M\le\varepsilon/3.$$

Finally, writing

$$\begin{aligned}\sum_q|a_q^nc_{qp}^n-a_qc_{qp}|\le&\sum_{q\le q_0}|a_q^nc_{qp}^n-a_qc_{qp}|\\&+\sum_{q>q_0}|a_q^nc_{qp}^n|+\sum_{q>q_0}|a_qc_{qp}|\end{aligned}$$

and using Proposition 3.2 and Point 2) we see that there exists n_2 so that

$$\sum_q|a_q^nc_{qp}^n-a_qc_{qp}|\le\varepsilon$$

for each $n \geq n_2$. Hence (4.1) is proved.
4) By combining Point 1) and Point 3), we then obtain

$$\lim_{n \to +\infty} \langle B_n f_n, A_n^{-1} e_p \rangle_n = \langle Bf, e_p \rangle.$$

This finishes the proof of Proposition 4.1. $\square$

The following proposition can be considered as a converse of Proposition 4.1.

Proposition 4.2. *Let ρ be a uB-contraction of (π_n). Assume that for each $f \in D$ there exists an integer $n(f)$ such that $\|R_n f\|_n = \|f\|$ for all $n \geq n(f)$. Then ρ is a MN-contraction of (π_n).*

Proof. Define the operators A_n by $A_n := \tilde{R}_n^{-1} : \mathcal{H}_n \to (\mathrm{Ker}\, \tilde{R}_n)^{\perp} \subset \mathcal{H}$. The additional assumption guarantees that each $f \in D$ lies in $A_n(\mathcal{H}_n)$ for n large enough. Moreover, for each $f \in D$, the sequence $A_n^{-1} f = R_n f$ converges to f and then the sequence $B_n R_n f$ converges to Bf (here we use the same notation as in the proof of Proposition 4.1). In other words, we have

$$\langle R_n g, B_n R_n f \rangle_n = \langle g, A_n B_n A_n^{-1} f \rangle \to \langle g, Bf \rangle$$

as $n \to +\infty$ for each $g \in D$. This shows that $(A_n B_n A_n^{-1} f)$ converges weakly to Bf in $\mathcal{H}$. Since $\|A_n B_n A_n^{-1} f\| = \|f\| = \|Bf\|$, we find that $(A_n B_n A_n^{-1} f)$ converges strongly to Bf. This gives the desired result. $\square$

We also have the following result.

Proposition 4.3. *If ρ is a nuMN-contraction of (π_n) then ρ is a B-contraction of (π_n).*

Proof. For each n, we fix a subspace $S_n \subset \mathcal{H}$ complement to $A_n(\mathcal{H}_n)$ and we define the operator R_n by $R_n = A_n^{-1}$ on $A_n(\mathcal{H}_n)$ and $R_n = 0$ on S_n. For each f and g in D, we have $f, g \in A_n(\mathcal{H}_n)$ for n large enough. Then, recalling Definition 2.7, we have

$$\langle R_n f, R_n g \rangle_n = \langle A_n^{-1} f, A_n^{-1} g \rangle_n \to \langle f, g \rangle$$

as $n \to +\infty$. Hence $(\mathcal{H}_n, R_n)$ is an inductive resolution of $\mathcal{H}$.

The rest of the proof goes as in the proof of Proposition 4.1. The only change is that we have $\lim_{n \to +\infty} \|A_n^{-1} e_p\|_n = 1$ instead of $\|A_n^{-1} e_p\|_n = 1$ for each n. $\square$

5. Examples, Remarks and Open Questions

5.1. **A contraction of $SU(1,1)$ to the Heisenberg group.** Here we take $G = SU(1,1)$ and H to be the 3-dimensional Heisenberg group. Let $\{X, Y, Z\}$ be a basis of $\mathfrak{h}$ in which the only nontrivial bracket is $[X, Y] = Z$. We consider the contraction of $\mathfrak{g}$ to $\mathfrak{h}$ defined by

$$C_r(aX + bY + cZ) = \frac{1}{2} \begin{pmatrix} -ir^2 c & r(b - ia) \\ r(b + ia) & ir^2 c \end{pmatrix}$$

and the corresponding group contraction defined by

$$c_r(\exp_H(aX+bY+cZ)) = \exp_G C_r(aX+bY+cZ).$$

The group G acts on the unit disk $\mathbb{D} = (|z| < 1)$ by linear fractional transformations. For each integer $n > 2$, we consider the Hilbert space $\mathcal{H}_n$ of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_n^2 := \int_{\mathbb{D}} |f(z)|^2 \, d\mu_n(z) < +\infty$$

where $d\mu_n(z) := \frac{n-1}{\pi}(1-|z|^2)^{n-2}\, dxdy$. Here we denote by $dxdy$ the Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$. The family $f_p^n(z) = \binom{n+p-1}{p}^{1/2} z^p$, $p \in \mathbb{N}$, is an orthonormal basis of $\mathcal{H}_n$. Let π_n be the unitary representation of G on $\mathcal{H}_n$ given by

$$\pi_n(g)\, f(z) = (a - \bar{b}z)^{-n}\, f(g^{-1}\cdot z), \qquad g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}.$$

The family (π_n) is then called the holomorphic discrete series of $G = SU(1,1)$ [21].

On the other hand, we fix a real number $\lambda > 0$ and we denote by $\mathcal{H}_\lambda$ the Hilbert space of all holomorphic functions f on $\mathbb{C}$ such that

$$\|f\|_\lambda^2 := \int_{\mathbb{C}} |f(z)|^2 \, d\mu_\lambda(z) < +\infty$$

where $d\mu_\lambda(z) := \frac{1}{2\pi\lambda}\exp(-|z|^2/2\lambda)\, dxdy$. The family

$$f_p^\lambda(z) = \frac{1}{\sqrt{(2\lambda)^p p!}}\, z^p, \qquad p \in \mathbb{N},$$

is an orthonormal basis of $\mathcal{H}_\lambda$. Let ρ_λ be the unitary irreducible representation of H on $\mathcal{H}_\lambda$ defined by

$$\begin{aligned}\rho_\lambda(&\exp_H(aX+bY+cZ))\, f(z) \\ &= \exp(ic\lambda + \tfrac{1}{4}(b+ai)(2z+\lambda(-b+ai))\, f(z+\lambda(-b+ai)).\end{aligned}$$

In [7], we showed that for each $\lambda > 0$, ρ_λ is a MN-contraction of (π_n) with $r(n) = \sqrt{2\lambda/n}$, the operators A_n being defined by $A_n f_p^n = f_p^\lambda$. Then ρ_λ is also a uB-contraction hence a B-contraction of (π_n).

Note that another choice of the operators A_n does not necessarily lead to a contraction result. For example, for each n, consider a unitary operator B_n from $\mathcal{H}_n$ onto $\mathcal{H}$ so that $B_n^{-1} f_0 = f_{n^2}^n$. Then for $h = \exp_H(cZ)$, $c \neq 0$, we have

$$\begin{aligned}\langle B_n \pi_n(c_{r(n)}(h)) B_n^{-1} f_0, f_0\rangle_n &= \langle \pi_n(c_{r(n)}) f_{n^2}^n, f_{n^2}^n\rangle_n \\ &= \exp\left(-icn\frac{r(n)^2}{2} + icn^2 r(n)^2\right)\end{aligned}$$

which does not converge as $n \to +\infty$. Thus $(B_n\pi_n(c_{r(n)}(h))B_n^{-1}f_0)$ does not converge in $\mathcal{H}$. Incidently, this also shows that the sequence $\|A_n\pi_n(c_{r(n)}(h))A_n^{-1} - \rho(h)\|_{\mathrm{op}}$ does not converge to 0 as $n \to +\infty$. Indeed, for $u_n = A_nB_n^{-1}f_0 \in \mathcal{H}$ we have

$$\begin{aligned}\|A_n\pi_n(c_{r(n)}(h))A_n^{-1}u_n - \rho(h)u_n\|^2 \\ &= 2 - 2\,\mathrm{Re}\langle \pi_n(c_{r(n)}(h))B_n^{-1}f_0, A_n^{-1}\rho(h)A_nB_n^{-1}f_0\rangle \\ &= 2 - 2e^{ic\lambda}\,\mathrm{Re}\langle \pi_n(c_{r(n)}(h))B_n^{-1}f_0, B_n^{-1}f_0\rangle\end{aligned}$$

because $A_n^{-1}\rho(h)A_nf(z) = \rho(h)f(z) = e^{ic\lambda}f(z)$ for each $f \in \mathcal{H}$. Hence $\|A_n\pi_n(c_{r(n)}(h))A_n^{-1}u_n - \rho(h)u_n\|$ does not converge to 0.

More generally, in [10], we obtained a MN-contraction of the discrete series of a semi-simple non-compact Lie group to the direct product of a Heisenberg group by an abelian group.

Similarly, in [6], we gave a MN-contraction of the unitary irreducible representations of $SU(2)$ to the representation ρ_λ (see also [23]). This result was partially generalized in [9].

5.2. **Some open questions.** Here we mention some problems and open questions about contractions of representations which are motivated in particular by the previous examples.

(1) Let G be a semi-simple non-compact Lie group. Assume that there is a group contraction of G to a Lie group H. What unitary representations of H are MN-contractions of representations of the unitary principal series and of the discrete series (if there exists) of G ?
(2) Let G be a semi-simple compact Lie group. Assume that there is a group contraction of G to a Lie group H. What unitary representations of H are contractions of the unitary irreducible (finite-dimensional) representations of G?
(3) The problem of the 'unitarization' of the contractions of representations. In the literature, we can find some examples of nuMN-contractions. For instance, a nuMN-contraction of the discrete series of $SU(1,1)$ to some unitary representations of $\mathbb{R}^{n+1} \rtimes SO(1,1)$ was given in [12]. Then, a natural question is whether such a nuMN-contraction is also a MN-contraction. More generally, is any nuMN-contraction of unitary representations a MN-contraction?

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